Knowing Where You Are Going *Helps You* Know How to Get There

PETER SULLIVAN, REBECCA YOUDALE and **ROBYN JORGENSEN** describe tasks that help facilitate students' understanding of subtraction.

Introduction and context

This is a report of a teaching exploration at an Indigenous Community School in the Kimberley region of Western Australia that sought to use a specific way of thinking about particular content domain, subtraction, to develop focused mathematically rich learning experiences. Peter was at the school as part of the Maths in the Kimberleys project, which is led by Robyn Jorgensen (Zevenbergen) from Griffith University, and is an Australian Research Council (ARC) project in partnership with the Association of Independent Schools of Western Australia (AISWA). A set of four 90 minute lessons was planned and taught by Rebecca, building on activities suggested by Rebecca and Peter, and incorporating the pedagogical approach being advocated by the project. There was an interview assessment of student learning after the lessons.

Effective teaching is multi-dimensional, and requires consideration of a variety of factors including: the culture and background of the children; the mathematics they already know, the mathematics they need to learn, the nature of the desired learning; the experiences that might support that learning; the pedagogies that can support the experiences; and the processes that can be used for assessing learning and evaluating teaching. In this context there are enormous

advantages to knowing where you are going, and how this helps you know when you are there.

First we explain the importance of having clear mathematical goals (knowing where you are going), and then we describe some illustrative activities (how to get there) that were used over the four lessons. We also present some data that indicate that there was positive student achievement as a result of the experience, as well as some further challenges.

The importance of having clear mathematical goals: The case of subtraction

The overall project is based on a hypothetical pedagogical framework derived from the work of Boaler (2008) and elaborated by Grootenboer (2009), Jorgensen (2009), and Sullivan (2009). One aspect of the pedagogical framework is the use of rich and purposeful tasks. Clearly, tasks are more likely to be purposeful if teachers know where they are going. Hattie and Timperley (2007), for example, reviewed a range of studies on the characteristics of effective classrooms and found that feedback was among the main influences on student achievement. The key elements of feedback are:

- "Where am I going?"
- "How am I going?" and
- "Where am I going to next?"

The implication is that it is best that the teacher knows where the students are going, can make decisions on expectations for performance, and has some sense of where the experiences are leading.

If a teacher is to give adequate feedback to students, the teacher needs to have goals for particular sets of experiences, and know how to assess whether a student has achieved these goals. It is, therefore, helpful if teachers are aware of the key learning ideas, can formulate appropriate interim goals, and provide interactive and on-going advice to students on their achievement of any interim goals. Indeed one of the critical challenges for those who work with teachers is to assist them in doing this.

In exploring what this might look like, we focused on the topic of subtraction. The topic was suggested by Rebecca who was teaching eleven Grade 3 and 4 students. Rebecca reported that, although the students had had a range of experiences involving number sequences and addition, their experience with subtraction was limited.

Our approach was based on an understanding that these children had already experienced preliminary ideas. These included: saying forward number sequences; ordering numbers; recognising that number of objects remains the same however they are arranged (conservation); that the last number counted is the number of objects (cardinality); and the recognition without counting, of the number of objects in a group (subitising).

The plan was that Rebecca would address four of the key initial aspects in learning subtraction that we hypothesise to be:

- stating the number 1 and 2 before a given number;
- modelling numbers in terms of their parts;
- using mental strategies for subtraction; and
- connecting different representations of subtraction.

It is not intended that these aspects represent a strict sequence, but that the four aspects together constitute this phase of learning subtraction, and can inform the selection of the tasks.

Choosing tasks that help achieve these learning goals

The principles adopted by the project that relate to the choice of rich and purposeful tasks are that the teacher will:

- choose activities in which the students

engage that provide the experiences leading to later discussion of the mathematical ideas;
- review the key ideas that are fundamental to the proposed experiences, and clarify any language required for appreciating the task, recording results, or reporting on findings;
- interact with students while they engage in the experiences, encouraging students to interact with each other including asking and answering questions, supporting students who need it, and challenging those who are ready;
- facilitate classroom discussions that build on the experiences and which draw from students their insights and discoveries; and
- explicitly summarise the key learning.

Our intention was to choose tasks and activities that could facilitate these actions. For this particular set of lessons, we selected activities that:
- had clear explicitly-outlined mathematical goals;
- had easily communicated instructions;
- allowed opportunities for pupils to make decisions; and
- provided a variety of representations, in this case, of subtraction.

In most cases, the activities were undertaken first as a whole class, emphasising the focus of the teaching, then in small groups, involving all students.

The following sections present an illustrative task or activity for each of the four aspects of learning subtraction suggested above.

Stating the number 1 and 2 before a given number

Familiarity with the sequence of numbers, in the absence of objects, is a prerequisite to counting. Saying the number sequence is also a prerequisite to perceptual counting, which is knowing the number of objects in a collection without having to count them one by one. Further, knowing “one more” and “two more” than a given number prepares students for addition. Likewise, familiarity with the numbers “one before” and “two before” a given number is connected to partitioning, and lays the foundation for subtraction.

The following is an example of an activity used to further this goal. The summary is written in the form of suggestions for other teachers. The indented text is an extract from the observer’s notes from the actual lessons, and there is a brief reflection on the activity:

Race to 0

Starting at 10, players take turns to take away either 1 or 2 from the previous total. The winner is the person who says 0. The following is an example of this game:

Player One	10		7		3		0
Player Two		8		5		2	

This game emphasises 1 before and 2 before. Another aspect of the game relates to the existence of a winning strategy. Rather than detracting from its effectiveness, the existence of the strategy enhances the search for mathematical connections. The observer noted:

> After some illustrative games, the pupils paired off and played together. The games provided further practice at 1 before and 2 before. Not all students were fluent, but there was plenty of opportunity for practice.

The game was collaborative, engaging and interactive. There was opportunity for a concluding review in which students reported back on their strategies. The activity is, of course, able to be varied by using different starting and target numbers. It seemed that students developed some familiarity with this notion of 1 and 2 before a given number.

Modelling numbers in terms of their parts

The earliest school learning should allow students to see that any number can be thought of as a whole and also in terms of its parts. Later students come to see that different ways of partitioning numbers are useful for different purposes. The focus at this stage was on activities that presented students with a whole, and one part, with the challenge being to find the other missing part. An illustrative activity is as follows:

One handful

Two students with one bag of counters. Have a bag that has, say, 15 unifix in it. One student takes a handful out and counts the counters. Without looking, the other student has to say how many are left in the bag. They can then count them together.

The purpose is for the pupils to work on the removed group (the handful) and use this to calculate what is left. The observer noted:

> This was done first as a whole class activity. It seemed that most students were engaged, they understood the purpose of the task, and they were prepared to persist to find an answer. It was noticeable that some students tried first to guess the answer, with the teacher's reaction being a clue to correctness. In each case Rebecca asked the student to explain their thinking. While many had some difficulty in doing this, it indicated that the emphasis was as much on their strategy as their answer.

This activity has simple rules. It involves group work including discussion, which in this case was often in Kriol, the teacher and Aboriginal Education Worker (AEW) interacted with the groups, it suited different levels of readiness, and there was a reporting back process which valued all of the methods of calculation.

Mental strategies that are useful for subtraction

At some stage, students start to work with numbers they are imagining, which are not necessarily connected to specific objects. This can only work if the students have some fluency with the numbers they are operating on, and perhaps confidence that they can devise a solution path for a particular calculation. For subtraction, we see the key facts that can be practised as:

- taking away 1 and taking away 2;
- building to 10 (6 + ? = 10, etc.);
- doubles (the idea that if 3 + 3 is known then 6 – 3 can be developed);
- subtracting 10; and
- recognising families as a way of simplifying calculations (if you know 19 – 2, you can work out 19 – 17).

The following is an example of an activity that focuses on an aspect of this:

The difference is 2: An open-ended investigation

The problem is posed:

I am thinking of two numbers. The difference between the numbers is 2. What might be the numbers?

This can be recorded symbolically as

$$\boxed{\_ - \_ = 2}$$

The point is that pupils can explore aspects of a difference of 2, and even recognise the patterns of difference that appear. It gives pupils the opportunity to make active decisions about the numbers they use and the way they record their results. It is important to emphasise that there is more than one possible answer, and that it helps if the answers are written systematically. The observer noted:

> The pupils worked productively on the task, and most groups were willing and

able to produce multiple solutions, some of which were systematically organised. Making choices seemed to be engaging. In this case there was a need for extended explanations of how this would work. Perhaps in the future it will be easier to pose such tasks. The students worked in groups with particular roles. Rebecca encouraged the reporter from each group to explain the process whereby the group found their particular set of answers. Again this reflects the focus on students explaining their strategies, supported by the teacher. The success of the groups is indicated in the diversity of responses given by one of the groups (see Figure 1).

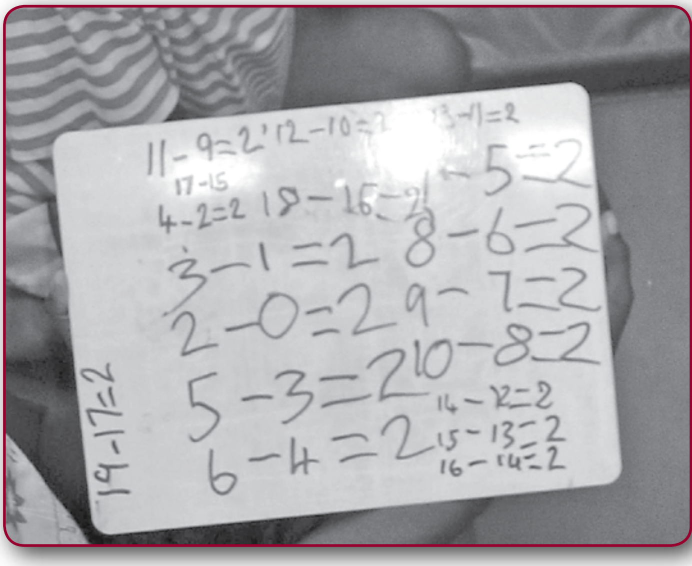

Figure 1. One groups' responses to the "difference is 2" task.

Note that this group, at least, has developed a range of possible solutions, appears to have identified a pattern in the solutions, and has laid the groundwork for readily knowing the answers to questions such as 19 – 17.

The pedagogies associated with this task are illustrative of the approach we are advocating. The task had a variety of entry levels, it can be answered in different ways, it involves group work with roles, and it allows a detailed and focused class discussion of strategies and patterns. The concluding review allows the teacher to highlight particular student insights.

Connecting different representations of subtraction

This phase of learning to subtract recognises that there are various forms in which problems can be presented, all of which are solved by one of the processes of subtraction. This is partly about the words such as subtract, take away, difference, change, and about the forms in which the problems can be posed. The important initial forms are as follows

$$12 - 3 = ?;\ 12 - 9 = ?;\ 12 - ? = 3;\ 12 - ? = 9$$

The following is an activity that provides a bridge to connecting different representations of subtraction:

Four representations

Each group has a set of 16 cards, with four different subtraction calculations represented, and 4 representations of each calculation, such as "12 – 7", "5", "Take 7 away from 12", and a drawing such as 12 birds with 7 flying away. A set of "four representations" cards is shown in Figure 2.

Figure 2. The "four representations" cards.

The cards can be used for structured games such as concentration, or snap, or placed on a prepared board. In this case, the students were asked to, in pairs, group the cards however they liked. The observer noted that;

> There was diversity in their responses, but it seemed that the students were able to decode the representations, and most of the groups, unsupervised, were able to suggest some possible groupings. Only one group achieved this by themselves, and another with help, grouping all four representations together. Interestingly, the cards were somewhat hastily created, yet the representations and the diversity of terms did not create difficulties. All groups were able to match up some different representations and explain the rationale for their choices. The focus in this case is representations of subtraction, although there is a need for the teacher to emphasise this purpose.

This activity allowed small group work on a meaningful and challenging task which facilitated the making of connections between representations. It fostered group discussions, and allowed summary discussions and reporting back from the groups.

Measuring pupils' performance on related items after the teaching

Short videos of the students' responses to these experiences indicated that they were engaged in the respective activities. To assess further the learning of these aspects of subtraction, the students were interviewed individually using the Victorian Early Numeracy Research Project interview (Clarke et al., 2001). The focus of the interview was on whether or not students had mastered particular levels within the number domains: counting and addition/subtraction were used in this case. The full interview covers place value, multiplication and division, and measurement and shape.

The success of the teaching was also evident in an examination of students' responses to particular items from the interview. For example, nearly all Grade 4 students and most Grade 3 students were able to:

- state the number sequence from 23 to 15, and from 10 to 0
- state the number before 56
- answer the question: "I have 8 biscuits, and I eat 3. How many do I have left?"
- answer: What is ten take away seven?

These elements were all part of the activities in the lessons.

On the other hand, for the item "I have 12 strawberries, and eat 9. How many are left?" only two Grade 4s and no Grade 3s could do this. There were at least two specific activities that focussed on these types of questions, and on using the simpler task (12 – 3) as a way of solving this more difficult form. This suggests that the next important learning goal for these students is to build connections between the related forms of the subtraction task; that is, if they know 12 – 3, then they can work out 12 – 9.

Summary

In her teaching, Rebecca used a clear and succinct set of hypothetical stages for the learning of subtraction to inform her planning and teaching decisions. She selected a range of interesting and engaging activities that included opportunities for students to make decisions. She supported her teaching by encouraging students to share their thinking. Most students learned most of the skills that had been the focus of instruction, but did not master the more difficult forms, even though this had been the focus of some activities. Nevertheless, the hypothetical stages appeared to be a useful way of thinking about subtraction.

The basic argument is that it is helpful for teachers to have clear mathematics goals (knowing where you are going) for the learning experiences they create and to be able to choose tasks (knowing how to get there) that prompt particular goal directed student activity, with the purpose of the task clear to students and teacher alike. A further aspect of effective teaching is to teach interactively, engaging the students in dialogue about their strategies and

approaches. The next challenge is finding ways to extend student thinking and build connections between representations.

References

Boaler, J. (2008). Promoting 'relational equity' and high mathematics achievement through an innovative mixed ability approach. *British Educational Research Journal, 34*(2), 167–194.

Clarke, D., Cheeseman, J., Gervasoni, A., Gronn, D., Horne, M., McDonough, A., Montgomery, P., Roche, A., Sullivan, P., Clarke, B. & Rowley, G. (2002). Early Numeracy Research Project (ENRP): Final report. Retrieved 11 September 2002 from: http://www.sofweb.vic.edu.au/eys/num/enrp.hmt.

Grootenboer, P. (2009). Rich mathematical tasks in the Maths in the Kimberley Project. In R. Hunter, B Bicknell & T. Burgess (Eds). *Crossing divides.* (Proceedings of the 32nd conference of the Mathematics Education Research Group of Australasia, Vol. 2, pp. 696–699). Palmerston North, NZ: MERGA.

Hattie, J. & Timperley, H. (2007) The power of feedback. *Review of Educational Research*, 77–(1), 81–112.

Jorgensen, R. (2009). Cooperative learning environments. In R. Hunter, B Bicknell & T. Burgess (Eds), *Crossing divides.* (Proceedings of the 32nd conference of the Mathematics Education Research Group of Australasia, Vol 1, pp. 700–703). Palmerston North, NZ: MERGA.

Sullivan, P. (2009). Describing teacher actions after student learning from rich experiences. In R. Hunter, B Bicknell, & T. Burgess (Eds.). *Crossing divides.* (Proceedings of the 32nd conference of the Mathematics Education Research Group of Australasia, Vol. 2, pp. 726–732). Sydney: MERGA.

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